Linear Stability of a Plasma Diode

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The stability of a plasma diode with respect to longitudinal oscillations is investigated. If there are free particles emitted by the electrodes, the perturbations do not have the same dynamics as they would in an infinite plasma, contrary to the case where only particles trapped in the diode are present. This can be interpreted as due to a coupling of plane waves of different wave lengths, introduced by the boundary conditions at the electrodes. The occurrence of resonant-particle effects, on the other hand, is subjected to precisely the same conditions as in an infinite plasma.

Most of the work done in the stability analysis of collisionless plasmas has dealt with infinite homogeneous or weakly inhomogeneous cases. Rather little has been done yet for finite plasmas, in particular for plasmas contained between electrodes or walls, while many laboratory plasmas are of this type. These have been mainly considered so far in the context of calculations of their response characteristics with respect to an external electric field, assuming that the plasma is stable. Both the case of a semi-infinite plasma confined by a single wall and that of a plasma diode has been treated (cf. e. g., [1–8]). Also situations where the plasma is confined by a particular space-dependent electrostatic potential have been investigated [9–11]. Whereas a general kinetic theory of the stability of inhomogeneous equilibria has been developed by BUNEMAN [12] and BERNSTEIN [13], an explicit solution has been derived only for the case of special inhomogeneous electron plasmas, relevant to the problem of the neutralization of ion beams, by ROSENBLUTH, PEARLSTEIN, and STUART, using a variational principle [14]. Besides, conclusive results about stability have been obtained only in the case of a plasma diode with perfectly reflecting walls, which can be shown to be equivalent to that of an infinite homogeneous plasma with special periodicity of the perturbation [15, 16] (cf. also Refs. [17]).

In this paper we investigate the problem of the stability of a plasma diode against electrostatic longitudinal oscillations, allowing for particle emission from, and absorption at, the electrodes. We use a kinetic treatment based on the Vlasov equation, supposing the particle densities sufficiently low for collisions to be negligible.

In Sect. I we define the model used to represent the plasma diode, i. e., the self-consistent equilibrium we start from. In Sect. II the corresponding particle orbits are given, and the particles are classified according to the kind of their motion. Sect. III contains the integration of the linearized collisionless Vlasov equation over the particle orbits according to the usual scheme, leading to explicit expressions for the first order distribution functions of the plasma particles. In Sect. IV we write the complete dispersion relation of the problem, bringing it into a

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form which displays the physical origin of the different terms. As this dispersion relation is very complex, it is impossible to solve it explicitly. Nevertheless, some information about its physical content can be gained. The structure of the terms coming from free particles (particles emitted into the system from, and reabsorbed by, the walls) and from trapped particles (reflected by the walls and staying in the system indefinitely) is clearly different. One is therefore naturally led to consider two auxiliary problems: a plasma diode containing only trapped particles and one in which all particles are free. These two cases are studied separately in Sect. V and VI, respectively. A comparison of them in relation to the case of an infinite homogeneous plasma reveals the different role played by the two kinds of particles and permits us to draw some conclusions about the situation in a real plasma diode.

I. Equilibrium

Let us consider a one-dimensional, finite system of length $2L$, bounded by two hot plates, kept in general at different potentials. Each plate emits particles into the plasma. As a consequence, a charged sheath will appear in front of each plate, whereas charge neutrality is maintained inside the plasma. As is well known, the thickness of the sheath is of the order of the Debye length of the plasma, which we assume much smaller than the dimension of the system and the relevant wave lengths of the perturbations to be considered.

Then we can ignore the width of the sheaths, and take a square profile for the electric potential, e. g., as represented in Fig. 1 (for the case of

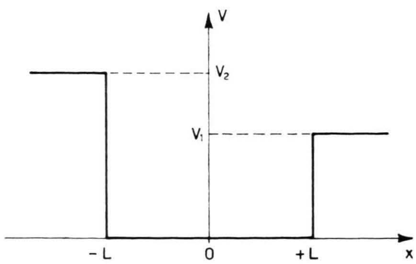

Fig. 1. Example of a potential distribution in a plasma diode (electron-rich sheaths).

electron-rich sheaths). Corresponding to this potential, one has zeroth order distribution functions $f_{j0}(v_x)$ of electrons ($j=e$) and ions ($j=i$) of the type represented in Fig. 2. The example plotted refers to the self-consistent equilibrium considered by SESTERO and ZANNETTI[18] where the distributions are made up of pieces of Maxwellians. For the wall po-

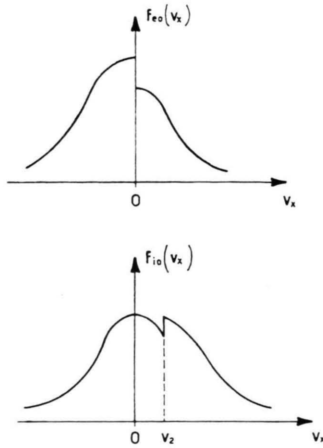

Fig. 2. Particle distribution functions in a plasma diode with electron-rich sheaths: a) electron distribution, b) ion distribution; v_2 is defined by Eq. (1.1a).

tentials V_1 and V_2 chosen, the distribution of the electrons is discontinuous at $v_x=0$, that of the ions at $v_x=v_2$, where v_2 is defined by

$$v_2 \equiv \sqrt{2\,e\,V_2/m_i}\,, \qquad (1.1a)$$

with e the elementary charge and m_i the ion mass. For reference, we introduce also

$$v_1 \equiv \sqrt{2\,e\,V_1/m_i}\,. \qquad (1.1b)$$

In the following we shall mainly refer to a situation of the kind just described. The generalization to an arbitrary square potential is straightforward.

II. Particle Orbits

We divide the plasma particles into two categories, calling *free particles* those which are emitted and reabsorbed by the walls (and therefore stay only for a finite time in the system), and *trapped particles* those which move back and forth between the electrodes indefinitely.

We then see that for the example of the square potential of Fig. 1 (electron-rich sheaths), the electrons are always accelerated towards the walls in the actual sheath regions, and are therefore all free in the above sense, i. e., an electron, emitted from a

[18] A. SESTERO and M. ZANNETTI, Nuovo Cimento **51 B**, 230 [1967].

wall, will traverse the system once and will then be absorbed by the opposite wall.

The situation is quite different for the ions, which, for some energies, can be reflected from the walls. We therefore have both free and trapped ions. A classification of the possible ion orbits can be best made using an $x-v_x$ diagram (Fig. 3). There are three essentially different classes of ion orbits:

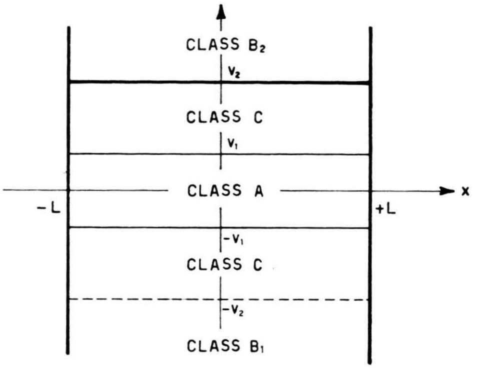

Fig. 3. Classification of ions according to their orbits for a plasma diode with electron-rich sheaths; v_1 and v_2 are determined by Eqs. (1.1).

Class A: $|v_x| < v_1$

These ions have not enough energy to cross either of the potential barriers at the walls. Therefore they are trapped and travel back and forth between the electrodes indefinitely.

Class B: $|v_x| \geqq v_2$

These ions have sufficient energy to pass through the potential barriers at both walls. Therefore this class is made up of ions which are emitted from one wall, traverse the system once, and are then absorbed by the opposite wall. The subclass B_1 contains particles emitted at $x=L$, the subclass B_2 those starting from $x=-L$.

Class C: $v_1 \leqq |v_x| < v_2$

These ions are emitted from the electrode at $x=+L$ with $v_x<0$. They traverse the system towards the electrode at $x=-L$, which is at a potential $V_2>V_1$, but do not have enough energy to reach it, hence they are reflected $(v_x>0)$, travel back, and are reabsorbed at $x=+L$.

The differences in the ion motions, expressed by this classification, are important when one wants to integrate Vlasov's equation with respect to time, in spite of the fact that the explicit analytical expression for the orbits is equal in all cases, namely

$$x(t') = x + v_x(t'-t) \tag{2.1}$$

(valid, of course, only for t and t' such that no reflection, emission or absorption occurs between the two time instants).

It may be noted that, in the sense of the above classification, the electrons, in the example considered, are all to be regarded as particles of class B.

III. First Order Distribution

We start from the linearized Vlasov equation for the perturbed distribution f_{j1} of particles of charge q_j and mass m_j

$$\frac{\partial f_{j1}}{\partial t} + v_x \frac{\partial f_{j1}}{\partial x} + \frac{q_j}{m_j{}^9} E_1 \frac{\partial f_{j0}}{\partial v_x} = 0\,. \tag{3.1}$$

For all the linearized quantities we introduce an exponential behaviour in time, e. g., for the electric field

$$E_1(x,t) = E_1(x)\, e^{i\omega t} \tag{3.2}$$

with $\mathrm{Im}(\omega) < 0$ to ensure convergence, as $t \to -\infty$, of all integrals to be evaluated in the following.

Then we integrate Eq. (3.1) along the (unperturbed) orbits in the usual way, considering separately the three classes of particles defined in the preceding Section. Thereby we do not specify, for the time being, the particle species nor the corresponding velocity intervals. Of course, in a linear approximation the dynamics of particles belonging marginally to one of the above classes is not treated properly, as far as the presence of the perturbation permits them to pass from one class into another. This means that their orbits are strongly perturbed even by a small field perturbation. The number of these "exceptional" particles decreases with the amplitude of the perturbation. It is a well-known condition for the validity of the linear approach that the over-all effect of such particles must be negligible.

A. Trapped particles

The formal solution of the Vlasov equation (3.1) is

$$f_{j1}(x,v_x) = -\frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\int_{-\infty}^{0} \mathrm{d}\tau\, E_1[x(\tau)]\, e^{i\omega t} \tag{3.3}$$

(having introduced the integration variable $\tau = t'-t$).

Since for trapped particles the orbit $x(\tau)$ is periodic with period

$$T = 4\,L/|\,v_x\,|\,, \tag{3.4}$$

it is convenient to introduce a Fourier series with respect to τ for E_1, i. e.,

$$E_1[x(\tau)] = \sum_n E_{1n}^{\pm}\, e^{i\cdot 2\pi n \tau/T} \tag{3.5}$$

with

$$E_{1n}^{\pm} = \frac{1}{T}\int_0^T E_1[x(\tau)]\, e^{-i\cdot 2\pi n\tau/T}\, d\tau\,, \tag{3.6}$$

where the superscript $\pm$ of E_{1n} refers to the sign of v_x. Then Eq. (3.3) yields

$$f_{j1}^{\pm}(x, v_x) = -\frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\sum_n \frac{E_{1n}^{\pm}(x)}{i[\omega + (\pi\, n/2\,L)\,|\,v_x\,|]}\,. \tag{3.7}$$

Upon introducing also spatial Fourier series for the linearized quantities, namely, for a given $h_1(x)$,

$$h_1(x) = \sum_p h_{1p}\, e^{ik_p x} \tag{3.8}$$

where

$$k_p = \pi\, p/\alpha\, L \quad \text{with } p \text{ integer}, \tag{3.9}$$

$$h_{1p} = \frac{1}{2\,\alpha\, L}\int_{-\alpha L}^{+\alpha L} h_1(x)\, e^{-ik_p x}\, dx\,, \tag{3.10}$$

we derive for either sign of v_x from Eqs. (3.6) and (3.7) [19]

$$f_{j\,1p}(v_x) = -\frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\sum_q \gamma_{qp}(v_x)\, E_{1q} \tag{3.11}$$

with

$$\gamma_{qp}(v_x) = \sum_n \frac{1}{i(\omega + \pi\, n\, v_x/2\, L)}\,\frac{\sin[\alpha\,\pi(p/\alpha - n/2)]}{2\,\alpha\,\pi(p/\alpha - n/2)}\left\{\frac{\sin[\pi(q/\alpha - n/2)]}{\pi(q/\alpha - n/2)} + (-1)^{n+1}\frac{\sin[\pi(q/\alpha + n/2)]}{\pi(q/\alpha + n/2)}\right\}. \tag{3.12}$$

In the preceding formulae, different choices for the parameter α correspond to using different sets of fundamental functions for the description of the space dependence of the perturbations. Correspondingly, the continuation of the physical quantities outside the physical interval $-L \leqq x \leqq L$ depends on α. E. g., for $\alpha = 1$ the period of the continued perturbation is $2\,L$, for $\alpha = 2$ it is $4\,L$. Since the only requirement for an expansion to be acceptable is that any physically possible perturbation of the form (3.2) can be represented, suitable constraints on the expansion coefficients can be introduced for values of $\alpha > 1$. Advantage can be taken of this liberty in the choice of the expansion (3.8) to simplify the dispersion relation and the eigenmodes of the problem (cf. Sect. V).

B. Free particles traversing the system once

According to Eq. (2.1) for the orbits, a free particle which is at the point x at a time t, moving with velocity v_x, has left the wall ($x = -L$ if $v_x > 0$, $x = +L$ if $v_x < 0$) at the time

$$t^{*\pm}(x, v_x, t) = t - (x \pm L)/v_x \tag{3.13}$$

where the upper sign and upper subscript correspond to positive velocities (class B_2), the lower to negative velocities (class B_1). The formal solution of the linearized Vlasov equation (3.1) is now

$$f_{j1}^{\pm}(x, v_x, t) = f_{j1}^{\pm}(\mp L, v_x, t^{*\pm}) - \frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\int_{t^{*\pm}}^{t} dt'\, E_1[x(t')]\, e^{i\omega t'} \tag{3.14}$$

where $f_{j1}^{\pm}(\mp L, v_x, t)$ represents the boundary conditions for $x = \mp L$ and $v_x \gtrless 0$, assumed to be of the form

$$f_{j1}^{\pm}(\mp L, v_x, t) = f_{j1}(\mp L, \mp|\,v_x\,|)\, e^{i\omega t}\,. \tag{3.15}$$

Introducing a Fourier representation according to Eqs. (3.8) to (3.10), Eq. (3.14) yields [19]

$$f_{j1}^{\pm}(v_x) = f_{j1}(\mp L, \mp|\,v_x\,|)\, e^{\mp i\omega L/v_x}\,\frac{\sin(\alpha\,\omega\, L/v_x + \pi\, p)}{\alpha\,\omega\, L/v_x + \pi\, p} - \frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\sum_q \alpha_{pq}^{\pm}(v_x)\, E_{1q} \tag{3.16}$$

with

$$\alpha_{qp}^{\pm}(v_x) = \frac{1}{i(\omega + \pi\, q\, v_x/\alpha\, L)}\left\{\delta_{qp} - \exp\left[\mp i\left(\frac{\omega\, L}{v_x} + \frac{\pi\, q}{\alpha}\right)\right]\frac{\sin(\alpha\,\omega\, L/v_x + \pi\, p)}{\alpha\,\omega\, L/v_x + \pi\, p}\right\}. \tag{3.17}$$

From Eq. (3.17) it is easily seen that for $\alpha = 1$ the matrix $\alpha_{qp}^{\pm}(v_x)$ is symmetric in q and p.

[19] Cf. M. DOBROWOLNY, F. ENGELMANN, and A. SESTERO, Internal Report LGI 68/24, Laboratori Gas Ionizzati, Frascati, Italy.

C. Free particles traversing the system twice

The calculation for free particles traversing the system twice proceeds analogously to that for the free particles which cross the system only once; some additional care must, however, be exercised.

In the example considered these particles are ions, starting from $x=+L$ with negative velocity $-v_2<v_x\leqq -v_1$. Thus if $v_x<0$ at the time t, the particle is on its first trip across the system. Correspondingly formula (3.16) with the lower signs, valid for class B_1, applies, that is for $-v_2<v_x\leqq -v_1$ one has

$$f_{j1p}^{-}(v_x)=f_{j1}(L,v_x)\,e^{i\omega L/v_x}\frac{\sin(\alpha\,\omega\,L/v_x+\pi\,p)}{\alpha\,\omega\,L/v_x+\pi\,p}-\frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\sum_q \alpha_{qp}^{-}(v_x)\,E_{1q}\,. \tag{3.18}$$

However, if $v_x>0$ at the time t, the particle is on its second trip across the system, having been reflected at $x=-\mathrm{L}$. Then the contributions from both trips must be taken into account, yielding

$$f_{j1p}^{+}(v_x)=f_{j1}(L,-v_x)\,e^{-3i\omega L/v_x}\frac{\sin(\alpha\,\omega\,L/v_x+\pi\,p)}{\alpha\,\omega\,L/v_x+\pi\,p}-\frac{q_j}{m_j}\frac{\partial f_{j0}}{\partial v_x}\sum_q\left[\beta_{qp}(v_x)+\alpha_{qp}^{+}(v_x)\right]E_{1q}\,. \tag{3.19}$$

where $v_1\leqq v_x<v_2$ and

$$\beta_{qp}(v_x)=-\frac{2\,L}{v_x}\,e^{-2i\omega L/v_x}\frac{\sin(\alpha\,\omega\,L/v_x+\pi\,p)}{\alpha\,\omega\,L/v_x+\pi\,p}\cdot\frac{\sin(\omega\,L/v_x-\pi\,q/\alpha)}{\omega\,L/v_x-\pi\,q/\alpha}\,. \tag{3.20}$$

In Eq. (3.19) the first term on the right-hand side represents the contribution of the first trip across the system, while the second term, connected with the second trip, parallels the expression that we obtained for the particles of class B_2.

IV. Dispersion Equation

If we insert the distribution function calculated in Sect. III into Poisson's equation

$$i\,k_p\,E_{1p}=4\,\pi\,e\int_{-\infty}^{+\infty}[f_{i1p}(v_x)-f_{e1p}(v_x)]\,\mathrm{d}v_x \tag{4.1}$$

and pay due attention, in the integrations over velocity, to the classification of particles given in Sect. II, we arrive at a system of linear, inhomogeneous equations for the Fourier components E_{1p} of the perturbed electric field. Therein the inhomogeneous terms are due to the boundary conditions at $x=\mp L$ and, hence, depend linearly on $f_{j1}(\mp L,\pm|v_x|)$.

Now two different cases are possible. First, if the boundary conditions (3.15) are imposed by external means, the above system determines, for any given frequency ω, directly the Fourier components E_{1p} as functions of the amplitude of the perturbation at the boundaries, and of p. Such a situation is to be considered when one wants to calculate the response characteristics of the plasma diode [4–8]. Secondly, in the absence of an external excitation, the boundary conditions may describe some coupling between the particle emission from the electrodes and the perturbation present in the plasma. Then $f_{j1}(\mp L,\pm|v_x|)$ can be expressed in terms of that perturbation. As a consequence, the above system of equations becomes homogeneous and yields a dispersion relation which determines the fundamental frequencies ω of the diode. This is the situation in which we are interested here.

The simplest case of this kind is the one where

$$f_{j1}(\mp L,\pm|v_x|)=0 \tag{4.2}$$

is valid, i. e., where the particle emission from the walls is unaffected by the perturbation of the plasma. Then Eqs. (3.11), (3.16), (3.18), (3.19), and (4.1) yield for the example of Sect. I

$$\sum_q[\delta_{qp}+A_{qp}]\,E_{1q}=0 \tag{4.3}$$

with

$$A_{qp}=\frac{4\,\pi\,e^2}{i\,k_p}\frac{1}{m_i}\left\{\int_{-v_1}^{+v_1}\mathrm{d}v_x\frac{\partial f_{i0}^{(A)}}{\partial v_x}\gamma_{qp}(v_x)+\int_{-\infty}^{-v_1}\mathrm{d}v_x\frac{\partial f_{i0}^{(B1)}}{\partial v_x}\alpha_{qp}^{-}(v_x)\int_{v_2}^{+\infty}\mathrm{d}v_x\frac{\partial f_{i0}^{(B2)}}{\partial v_x}\alpha_{qp}^{+}(v_x)\right.$$

$$\left.+\int_{v_1}^{v_2}\mathrm{d}v_x\frac{\partial f_{i0}^{(C)}}{\partial v_x}[\beta_{qp}(v_x)+\alpha_{qp}^{+}(v_x)]\right\}+\frac{4\,\pi\,e^2}{i\,k_p}\frac{1}{m_e}\left\{\int_{-\infty}^{0}\mathrm{d}v_x\frac{\partial f_{e0}^{(B1)}}{\partial v_x}\alpha_{qp}^{-}(v_x)+\int_{0}^{+\infty}\mathrm{d}v_x\frac{\partial f_{e0}^{(B2)}}{\partial v_x}\alpha_{qp}^{+}(v_x)\right\} \tag{4.4}$$

where superscripts have been added to the distribution functions f_{j0} to indicate the origin of the various terms according to the classification of Sect. II. In Eq. (4.4) the discontinuities of the functions $f_{j0}{}^{(\mu)}$ at the limits of the range of integration have to be treated as occurring *inside* this range.

The requirement that the system (4.3) have nontrivial solutions, yields the dispersion relation

$$\mathrm{Det} \| \delta_{qp} + A_{qp} \| = 0 . \tag{4.5}$$

The matrix in Eq. (4.5) is infinite and nondiagonal. Solving Eq. (4.5) is a rather formidable problem (even from the numerical point of view). In order to throw some light on its content, we shall in the following consider separately the contribution from trapped particles and that from free particles; more precisely, we shall discuss two auxiliary problems: 1) a plasma diode with trapped particles only, 2) a plasma diode in which only free particles, crossing the system once, are present.

As a last step, we shall then extrapolate the conclusions reached in the above two auxiliary problems to a plasma diode containing all types of particles.

V. Diode where all Particles are Trapped

Let us consider a situation in which a plasma diode contains only trapped particles. This would require, e. g., infinitely strong space charge double-sheath in front of the electrodes, reflecting all ions and electrons. Hence, this case is identical with that investigated by MONTGOMERY and GORMAN [15]. Their results can be easily re-derived from the preceding general formulation.

As a consequence of the reflection of the particles at the walls, the current is zero at $x = \mp L$; therefrom and from Ampere's law it follows that one has, for the perturbed electric field, the condition

$$E_1(x = \pm L) = 0 , \tag{5.1}$$

to be satisfied by the Fourier series (3.8) for $E_1(x)$. A particularly simple formulation is obtained by choosing the parameter $\alpha = 2$ and specializing Eq. (3.8) for $E_1(x)$ to

$$E_1(x) = \sum_{p>0,\,\mathrm{odd}} 2\, E_{1,\,p} \cos \frac{p\,\pi\,x}{2\,L} + \sum_{p>0,\,\mathrm{even}} 2\, i\, E_{1p} \sin \frac{p\,\pi\,x}{2\,L} \tag{5.2}$$

with E_{1p} being real (purely imaginary) for p odd (even). In this expansion every term fulfills condition (5.1) separately. On the other hand, the form of the series (5.2) does not impose any further restriction (apart from square-integrability) on the perturbation $E_1(x)$.

With $\alpha = 2$ and Eq. (5.2) for $E_1(x)$, formula (3.11) for the first-order distributions becomes

$$f_{j\,1p}(v_x) = - \frac{q_j}{m_j} \frac{\partial f_{j0}}{\partial v_x} \frac{E_{1p}}{i(\omega + \pi\, p\, v_x/2\,L)} , \tag{5.3}$$

which coincides with the corresponding relation for an infinite homogeneous plasma with the specification

$$k = k_p = \pi\, p/2\,L . \tag{5.4}$$

Consequently, also the dispersion relation following from Eq. (4.1) is the same as in the case of an infinite homogeneous plasma. The fundamental modes of the problem are identical with the single terms of the expansion (5.2), i. e., they are standing plane waves, built up as a superposition of two waves with equal wave lengths, propagating in opposite directions.

One therefore concludes: The fundamental modes of a diode containing only trapped particles are a subclass of those of an infinite homogeneous plasma which has the same particle distribution functions, the admissible wave lengths forming a discrete set. The corresponding fundamental frequencies coincide in both problems. In particular, if the infinite case is stable, this holds also for a plasma "trapped" between walls. These are the essential results of Ref. [15].

VI. Diode Containing only free Particles

We consider now a second auxiliary problem in which we have a plasma diode containing only free particles crossing the system once. This would be true for an equilibrium in which the potential walls in front of the electrodes are absent, so that the potential between the plates (emitting and absorbing plasma particles) is constant, which is admittedly a rather artificial situation. It allows us, however, to exemplify the role of free particles in a finite system.

In this case the dispersion relation (4.5) reduces to

$$\mathrm{Det} \| \varepsilon(k_q, \omega)\, \delta_{qp} + F_{qp} \| = 0 \tag{6.1}$$

where

$$\varepsilon(k_q, \omega) = 1 - \frac{1}{k_q} \sum_j \frac{4\,\pi\, q_j^2}{m_j} \int_{-\infty}^{+\infty} \mathrm{d}v_x \frac{\partial f_{j0}/\partial v_x}{(\omega + k_q\, v_x)} , \tag{6.2}$$

is the (longitudinal) dielectric constant of an infinite homogeneous plasma and

$$F_{qp} = \frac{1}{i\,k_p} \sum_j \frac{4\,\pi\,q_j^2}{m_j} \left[\int\limits_{-\infty}^{0} \mathrm{d}v_x \,\frac{\partial f_{j0}}{\partial v_x}\,\tilde{\alpha}_{qp}^{-}(v_x) + \int\limits_{0}^{+\infty} \mathrm{d}v_x \,\frac{\partial f_{j0}}{\partial v_x}\,\tilde{\alpha}_{qp}^{+}(v_x) \right] \quad (6.3)$$

with

$$\tilde{\alpha}_{qp}^{\pm}(v_x) \equiv \alpha_{qp}^{\pm}(v_x) - \frac{\delta_{qp}}{i(\omega + k_q\,v_x)} = -\frac{\exp\{\mp i(\omega\,L/v_x + \pi\,q/\alpha)}{i(\omega + \pi\,q\,v_x/\alpha\,L)} \, \frac{\sin(\alpha\,\omega\,L/v_x + \pi\,p)}{(\alpha\,\omega\,L/v_x + \pi\,p)} . \quad (6.4)$$

The dispersion matrix is, hence, infinite and nondiagonal, nor can it be diagonalized by simple combinations of propagating plane waves as in the case of a diode containing only trapped particles. Nevertheless, some general features, and especially, the relations between the present problem and the case of an infinite homogeneous plasma can be displayed.

First of all, in the limit $L \to \infty$ the expression (3.16) for $f_{j1p}^{\pm}(v_x)$ in terms of E_{1q} goes over into that valid for an infinite homogeneous plasma, provided that α is chosen equal to 1. Correspondingly, the dispersion relation (6.1) takes the form

$$\varepsilon(k_q, \omega) = 0 . \quad (6.5)$$

The contributions due to the finiteness of the system tend to zero as $1/L$ when $L \to \infty$.

For finite L, the fundamental modes of the problem are no longer plane waves of given wavelength as for an infinite homogeneous plasma, and a plasma diode containing only trapped particles, but a superposition of an infinite number of them. In the dispersion relation (6.1) the interaction between different propagating plane waves is described by the off-diagonal elements of the matrix F_{qp}. This interaction is due to the boundary conditions on the electrodes. In fact, if we compare our boundary value formulation (3.14) of the free-particle problem with the usual initial value formulation for an infinite system, for which

$$f_{j1}(x, v_x, t) = f_{j1}(x - v_x\,t, v_x, 0) - \frac{q_j}{m_j} \frac{\partial f_{j0}}{\partial v_x} \int\limits_{0}^{t} \mathrm{d}t'\, E_1[x(t'), t'] \quad (6.6)$$

with $f_{j1}(x,\ v_x,\ 0)$ being the perturbation at time $t = 0$, we see that the former can be re-formulated in terms of the latter, taking

$$f_{j1}(x, v_x, 0) = f_{j1}^{\pm}\left(\mp L, v_x, -\frac{x \pm L}{v_x}\right) + \frac{q_j}{m_j} \frac{\partial f_{j0}}{\partial v_x} \int\limits_{0}^{-(x \pm \bar{t})/v_x} \mathrm{d}t'\, E_1[x(t'), t'] \quad (6.7)$$

for $v_x \gtrless 0$. The particular boundary condition (4.2) corresponds to an initial condition

$$f_{j1}(x, v_x, 0) = -\frac{q_j}{m_j} \frac{\partial f_{j0}}{\partial v_x} \int\limits_{0}^{-(x \pm \bar{t})/v_x} \mathrm{d}t'\, E_1[x(t'), t'] \quad \text{for } v_x \gtrless 0 . \quad (6.8)$$

Hence, this contributes to the dispersion matrix and, because of its double dependence on x through $E_1(x)$ and the upper limit of the integral, introduces off-diagonal matrix elements.

As a general rule, the influence of the finiteness of the system is small if

$$\bar{t} \equiv L/|v_x| \gg 1/\gamma \quad (6.9)$$

holds for typical particle velocities, $\gamma = -\operatorname{Im}\omega$ being the gowth rate of the mode [cf. Eqs. (3.17) and (6.1) to (6.4) with $\alpha = 1$]. Since $\bar{t}$ is the transit time of a particle of velocity v_x, condition (6.9) is satisfied if, for a typical particle, this time is large compared with the growth time of the instability. If condition (6.9) is violated, there are always groups of plane waves which strongly couple, and the dispersion relation is modified with respect to the case of an infinite plasma.

Let us now investigate the occurence of resonant effects due to free particles, concentrating on the diagonal term in Eq. (3.17) which is the only one surviving in the limit $L \to \infty$. This can be done along the lines discussed for the initial value problem in an infinite plasma in the Appendix. One observes from Eq. (3.17) that, for finite L, $\alpha_{pp}^{\pm}(v_x)$ and, hence, $f_{j1p}^{\pm}(v_x)$ are regular for

$$v_x \to -\omega/k_p = -\omega\,L/\pi\,p ,$$

where again $\alpha = 1$ has been taken. If one associates, as is usually done, resonant effects with the occurrence of a pole in $f_{j1p}^{\pm}(v_x)$, one would be led to conclude that for finite L resonant effects are absent. As shown in the Appendix for the initial value problem in an infinite plasma, this argument is, however, not conclusive. The situation is, in fact, completely analogous in both cases.

Defining as resonant those free particles which travel less than half a wave length with respect to the frame of reference moving with a given wave during the time $\bar{t}$, i. e., for which

$$|(k_p v_x + \beta)\bar{t}| < \pi \quad \text{with} \quad \beta = \operatorname{Re}\omega \tag{6.10}$$

holds, one obtains, e. g., for the mean variation of their kinetic energy density,

$$\overline{\left(\frac{\mathrm{d}W_j}{\mathrm{d}t}\right)}_{\text{res}} = -\frac{q_j^2\,|E_{1p}|^2}{m_j}\int_{\text{res}} v_x\,\frac{\partial f_{j0}}{\partial v_x}\,\alpha_{pp}^{\pm}(v_x)\,\mathrm{d}v_x + \text{c.c.}$$
$$\approx -2\,\frac{q_j^2\,|E_{1p}|^2}{m_j}\,\bar{t}\int_{\text{res}} v_x\,\frac{\partial f_{j0}}{\partial v_x}\,\mathrm{d}v_x \tag{6.11}$$

upon assuming $\bar{t} \ll 1/\gamma$ and expanding $\alpha_{pp}^{\pm}$ (v_x) as given by Eq. (3.17) for $|(k_p v_x + \omega)\,\bar{t}| \ll 1$. The width of the resonant velocity range is now, according to Eq. (6.10),

$$\Delta v_x = \pi / k_p\,\bar{t}, \tag{6.12}$$

so that one obtains for

$$\Delta v_x \ll |\beta/k_p|\,,\, v_{tj}$$

with v_{tj} characterizing the scale of variation of $f_{j0}(v_x)$ around $v_x = -\beta/k_p$, or equivalently for

$$\bar{t} = L\,k_p/|\beta| \gg 1/|\beta|\,,\, 1/k_p\,v_{tj} \tag{6.13}$$

the estimate

$$\overline{\left(\frac{\mathrm{d}W_j}{\mathrm{d}t}\right)}_{\text{res}} \approx -2\,\pi\,\frac{q_j^2\,|E_{1p}|^2}{m_j}\,\frac{\beta}{k_p^2}\left(\frac{\partial f_{j0}}{\partial v_x}\right)_{v_x=-\beta/k_p}. \tag{6.14}$$

This result coincides with the variation of the energy density of resonant particles as calculated from the pole of $f_{j1p}^{\mp}$ (v_x) in the limit $L \to \infty$. Hence, also free particles cause resonant effects, provided that condition (6.13) is satisfied. The meaning of this condition is completely analogous to that of relation (A.11) in the case of the initial value problem for an infinite plasma: in passing through the system, the number of resonant particles, which decreases with the distance from the emitting electrode, must become small compared to the total number of particles emitted. Finally it may be noted that the first part of condition (6.13) can be written also

$$p \gg 1;$$

it is, hence, fulfilled for all plane waves of wavelength small compared to L.

Finally, it may be noted that the essential results of this Section can be easily shown to apply also in the presence of free particles traversing the diode twice.

Conclusions

The preceding analysis shows that the stability properties of a plasma diode may differ significantly from those of an infinite plasma, having the same velocity distributions of the particles. This is mainly due to a coupling between plane waves of different wavelength, introduced by the presence of "free" particles, emitted into the system from the electrodes, for which given boundary conditions are to be satisfied. However, if the growth time of an instability is short compared with the transit times of the bulk of the free particles moving through the system, this coupling is weak and the dynamics of the modes is approximately equal to that in the corresponding infinite system. As to the resonant-particle effects, these occur in a finite system under precisely the same conditions as in an infinite plasma: the duration of the interaction with the wave must be long enough for the number of resonant particles to be small compared to the total number of particles present.

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Appendix:

Resonant Particle Effects in an Infinite Plasma

For comparison we consider in this Appendix how, and under what circumstances, resonant particle effects arise in an infinite plasma, when there is an initial perturbation at time $t = 0$. Thereby, we strongly rely on the work on Landau damping by DAWSON [20], STIX9 [21], and O'NEIL [22].

For simplicity, we limit ourselves to a single longitudinal mode, whose electric field is

$$E_1(x,t) = 2\,E_0\cos(k\,x + \omega\,t)$$
$$= E_0\,e^{i(kx+\omega t)} + \text{c.c.} \tag{A.1}$$

and assume that within a certain range of velocities, to be identified later with the resonant domain, the initial perturbation of the distribution function of the species j of plasma particles vanishes, so that there, as a consequence of the linearized Vlasov equation (3.1),

$$f_{j1}(x, v_x, t) = -\frac{q_j\,E_0}{m_j}\,\frac{\partial f_{j0}/\partial v_x}{i(k\,v_x + \omega)}$$
$$\cdot\,[1 - e^{-i(\omega + k v_x)t}]\;e^{i(kx+\omega t)} + \text{c.c.} \tag{A.2}$$

[20] J. DAWSON, Phys. Fluids 4, 869 [1961].
[21] T. H. STIX, The Theory of Plasma Waves, McGraw-Hill Book Co., Inc., New York 1962, pp. 132–136.
[22] TH. O'NEIL, Phys. Fluids 8, 2255 [1965].

holds. The resonant velocity domain is physically defined by those particles which, during the time t, have travelled less than half a wavelength with respect to a frame of reference moving with the wave, so that they have been subjected to a non-oscillating field. This condition can be expressed by

$$|(k v_x+\beta)\, t|<\pi \quad \text{with } \beta=\operatorname{Re}\omega\,. \tag{A.3}$$

Let us consider, e. g., the mean variation of the kinetic energy density of the particles due to the presence of the wave

$$\overline{\left(\frac{\mathrm{d}W_{\mathrm{j}}}{\mathrm{d}t}\right)}=\frac{1}{\lambda}\int_{x}^{x+\lambda}\mathrm{d}x'\int \mathrm{d}v_x\, q_{\mathrm{j}}\, E_1(x',t)\; v_x\, f_{\mathrm{j}1}(x',v_x,t) \tag{A.4}$$

where $\lambda=2\,\pi/k$. With Eqs. (A.1) and (A.2) this yields

$$\overline{\left(\frac{\mathrm{d}W_{\mathrm{j}}}{\mathrm{d}t}\right)}=-\frac{q_{\mathrm{j}}^2\,|E_0|^2}{m_{\mathrm{j}}}\int\frac{v_x\cdot(\partial f_{\mathrm{j}0}/\partial v_x)}{i(k\,v_x+\omega)}\cdot[1-e^{-i(\omega+kv_x)t}]\,\mathrm{d}v_x+\text{c.c.} \tag{A.5}$$

Usually the term containing $e^{i(\omega+kv_x)t}$ is neglected for large t. This can be justified by deforming the contour of the v_x integration into a Landau path (cf., e. g., [23]). Then the remaining integrand in Eq. (A.5) has a resonant character for $v_x\to-\omega/k$ and the contribution of the corresponding pole to the integral is identified with resonant particle effects. If $|\gamma|=|\operatorname{Im}\omega|$ is small, this yields

$$\overline{\left(\frac{\mathrm{d}W_{\mathrm{j}}}{\mathrm{d}t}\right)}_{\mathrm{res}}=-2\,\pi\,\frac{q_{\mathrm{j}}^2\,|E_0|^2}{m_{\mathrm{j}}}\,\frac{\beta}{k^2}\left(\frac{\partial f_{\mathrm{j}0}}{\partial v_x}\right)_{v_x=-\beta/k} \tag{A.6}$$

for the variation of the energy density of resonant particles.

More physical insight is, however, achieved by calculating the variation of the energy density of resonant particles from the complete expression of Eq. (A.5) and the physical definition (A.3) of resonant particles. In this context it is important to emphasize that the integrand of Eq. (A.5), as it stands, has *no pole* at $v_x=-\omega/k$, so that its introduction and use for describing resonant particle effects might seem a mathematical artifice. Assuming that $|\gamma|\,t\ll 1$ is also valid, Eq. (A.5) yields, expanding $e^{i(\omega+kv_x)t}$ for $|(\omega+k\,v_x)\,t|\ll 1$,

$$\overline{\left(\frac{\mathrm{d}W_{\mathrm{j}}}{\mathrm{d}t}\right)}_{\mathrm{res}}\approx-2\,\frac{q_{\mathrm{j}}^2\,|E_0|^2}{m_{\mathrm{j}}}\;t\int_{\mathrm{res}} v_x\,\frac{\partial f_{\mathrm{j}0}}{\partial v_x}\,\mathrm{d}v_x\,. \tag{A.7}$$

Since the width of the resonant range follows from Eq. (A.3) to be

$$\Delta v_x=\pi/k\,t\,, \tag{A.8}$$

one obtains from Eq. (A.7) for

$$\Delta v_x\ll|\beta/k|\,,\; v_{t\mathrm{j}}\,, \tag{A.9}$$

where $v_{t\mathrm{j}}$ characterizes the scale of variation of $f_{\mathrm{j}0}(v_x)$ around $v_x=-\beta/k$, the estimate

$$\overline{\left(\frac{\mathrm{d}W_{\mathrm{j}}}{\mathrm{d}t}\right)}_{\mathrm{res}}\approx-2\,\pi\,\frac{q_{\mathrm{j}}^2\,|E_0|^2}{m_{\mathrm{j}}}\,\frac{\beta}{k^2}\left(\frac{\partial f_{\mathrm{j}0}}{\partial v_x}\right)_{v_x=-\beta/k} \tag{A.10}$$

which is identical with the result of Eq. (A.6). This shows that the somewhat formal introduction of resonant effects in the usual mathematical procedure is, in fact, equivalent to considering the effects of resonant particles as defined by Eq. (A.3), provided that

$$t\gg 1/|\beta|\,,\; 1/k\,v_{t\mathrm{j}} \tag{A.11}$$

holds. According to Eq. (A.8), the number of resonant particles decreases with time, for $t=0$ all particles being resonant. Condition (A.11) ensures that t is large enough for the number of resonant particles to become small compared to the total particle number. Of course, the preceding linear analysis is limited, on the other hand, to times t smaller than the oscillation period of particles trapped in the wave [22].

[23] I. B. Bernstein and F. Engelmann, Phys. Fluids **9**, 937 [1966].